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The Weber Point can be Found in Linear Time for Points in Biangular Configuration

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Abstract

The Weber point of a given point set P is a point in the plane that minimizes the sum of all distances to the points in P. In general, the Weber point cannot be computed. However, if the points are in specific geometric patterns, then finding the Weber point is possible. We investigate the case of *biangular configurations*, where there is a center and two angles α and β such that the angles w.r.t. the center between each two adjacent points is either α or β, and these angles alternate. We show that in this case the center of biangularity is the Weber point of the points, and that it can be found in time linear in the number of points.

1 Introduction

Consider the following optimization problem: given the locations of a set of customers, find a placement for a manufactory that minimizes the sum of the distances from all customers. This problem is known as *Weber's problem*, and can be formalized as follows: Given a set of points $P = \{p_1, \ldots, p_n\}$ and a point x in the plane, we define the **Weber distance** between x and P by $WD_P(x) := \sum_{p \in P} |p - x|$, where $|p - x|$ denotes the Euclidean distance between p and x. A point w is **Weber point** of point set P if it minimizes the Weber distance between P and any point x in the plane, i.e., if $WD_P(w) = \min_{x \in \mathbb{R}^2} WD_P(x)$. Thus, a Weber point minimizes the sum of all distances to the points in P.

Weber points have been studied for a long time. The first formulation of the problem for $n = 3$ is by Fermat (1600). Then it was studied under different assumptions: by Cavalieri (1647, three points vertices of a triangle); Fagnano (1775, $n = 4$); Tedenat (1810); Steiner (1837) [1]. However, Weber was probably the first who stated this problem with the purpose of minimizing the sum of the transportation costs from the plant to sources of raw material and to the market center: hence, this problem for n points has become known as the *generalized Weber* problem [9].

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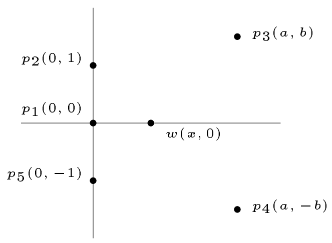

Figure 1: Example for intractable 5 points configuration [4]

If the points in P are on a line and the number of points n is even, then point w is Weber point of P if and only if it is positioned between the two median points, inclusive (note that in this case the Weber point is not unique). On the other hand, if n is odd, then the median point is the unique Weber point of P. Hence, for collinear points the Weber point(s) are known.

If the points are not collinear, then a Weber point for the points exists, and it is unique [2]. In the following, we will always assume that the points are not collinear (hence, $n \geq 3$).

Observe that in general, a Weber point is distinct from the center of gravity, defined by $p_{grav} := \frac{1}{n}\sum_{i=1}^{n} p_i$, that minimizes the sum of squares of distances to all points in P. The center of gravity, in contrast to the Weber point, is not invariant under straight line movement towards it. For a more detailed discussion on the differences between Weber points and centers of gravity, see [8].

No finite algorithm can exist that finds the Weber point of any arbitrary set of points, since finding the Weber point is equivalent to solving polynomial equations of high degree (which is known to be intractable). In particular, already for the simple configuration of 5 points shown in Figure 1 the Weber point cannot be computed [4]. However, several approximation algorithms can be found in literature. For instance, in [10], an intersting iterative geometrical construction to approximate the region of the plane where the weber point is, is presented. Other approximation algorithms for this problem have been presented in [7, 6].

On the other hand, in some point configurations it is indeed possible to determine the Weber point. In fact, it might be possible to find the roots of the first derivative of $WD_P(x)$, thus candidates for the Weber point. Moreover, in some special geometric patterns the Weber point can be characterized immediately. For instance, if there are two (or more) symmetry axes for the points, then the Weber point is at the intersection of these axes, since it is unique. Moreover, if the points are rotational symmetric, then the Weber point is at the center of rotational symmetry. Observe that the example in Figure 1 already implies that finding the Weber point is hard for some special geometric patterns, e.g. if there is only one axis of symmetry, or if the points are on a circle (by moving the point $(0, 0)$ to

In this paper, we present and investigate another geometric pattern where

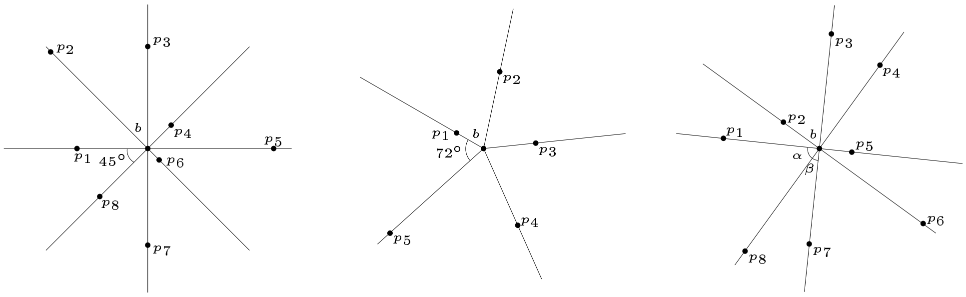

Figure 2: Example for a) equiangular configuration ($n = 8$) b) equiangular configuration ($n = 5$) c) biangular configuration ($n = 8$)

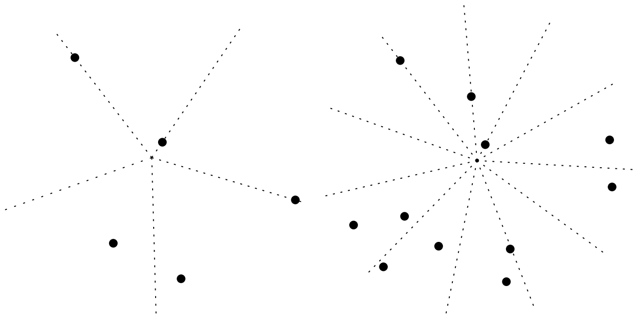

Figure 3: Examples for points not in equiangular configuration ($n = 5$ and $n = 11$)

the Weber point can be found in linear time, namely biangular point configurations. Here, we say that n points $P = \{p_1, \ldots, p_n\}$ are in *biangular configuration* if there exists a point $b \notin \{p_1, \ldots, p_n\}$ — the *center of biangularity* — an ordering of the points, and two angles[1] $\alpha, \beta > 0$ such that each two adjacent points form an angle α or β w.r.t. b, and the angles alternate (see Figure 2.c). In the special case where the two angles α and β are equal, we say that the points are in *equiangular configuration* (see fig. 2.a, 2.b). Figure 3 shows that most sets of points in the plane are not biangular.

If the points are in bianguluar configuration and the two angles α and β are different, then by definition n is even, since the angles alternate. In contrast, n can be even or odd if the points are in equiangular configuration ($\alpha = \beta$).

Biangularity is in a way a generalization of rotational symmetry: Assume that the configuration is biangular with center b, and in addition all points are on a circle with center b. Then the configuration is rotational symmetric as well. Hence, b is the Weber point of these points. Obviously, moving any of the

[1] All angles are oriented, say counterclockwise.

points straight in the direction of the center of biangularity b does not change the angles w.r.t. b, and the points remain in biangular configuration with the same center.[2]. Hence, the center of biangularity is always the Weber point of the points.

As mentioned before, Weber points have many applications in economics and social systems. However, our interest arises from the context of robotics, namely the problem of coordinating a set of "stupid" autonomous mobile robots that cannot communicate at all and that can only observe the positions of all other robots. Their task is to gather at any arbitrary point in the plane that is not fixed in advance. Although Weber points could - in principle - serve as meeting points, they do not solve the problem since they cannot be computed for any point set. Recently, biangular configurations have been applied successfully to gather the robots: in many initial configurations of the robots, they can first change their positions such that the configuration becomes biangular, and then they gather at the center of biangularity (which is in fact the Weber point). For a more detailed discussion, see [3].

The remainder of the paper is organized as follows. In Section 2 we present some basic properties of Weber points and biangular configurations. Then we present the linear time algorithm to find the center of biangularity (if any) in Section 3. Finally, we draw conclusions in Section 4.

2 Biangular Configurations and Weber Points

In this section, we present some basic properties of biangular configurations and Weber points, and in particular their connection.

First, observe that if the point configuration of n points $P = \{p_1, \ldots, p_n\}$ is rotational symmetric with rotation center c, then c is the Weber point of P: To see this, let δ be the angle of rotational symmetry. Let p'_i be the point arising from p_i after rotation by δ around c, for $1 \leq i \leq n$. Let w be the Weber point of $\{p_1, \ldots, p_n\}$, and w' the point arising from w after the rotation. Then w' is the Weber point of $\{p'_1, \ldots, p'_n\}$, since all points were rotated by the same angle. Since the points are rotational symmetric, we have $\{p_1, \ldots, p_n\} = \{p'_1, \ldots, p'_n\}$. Then $w = w'$, since the Weber point is unique. Obviously, the center of rotation c is the only point that remains invariant under rotation around c, thus the Weber point w must be the center of rotation c.

Using this, it is easy to see that if n points are in biangular configuration, then the center of biangularity is their Weber point, and hence unique.

Lemma 1. *Given a set of n points $P = \{p_1, \ldots, p_n\}$. If the points are in biangular configuration with centerpoint b, then b is the Weber point of P. Moreover, the center of biangularity b and the angles of biangularity α and β are unique.*

[2]It is even possible to relax the definition of biangular configurations and allow $b \in \{p_1, \ldots, p_n\}$. In this case, all adjacent points form angles α or β and they alternate, except for some gaps from the points that are at b. However, for sake of simplicity we will not consider this special case.

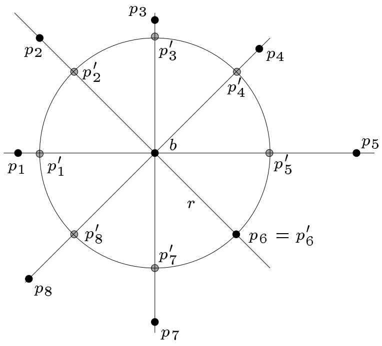

Figure 4: Constructing a circle from $p_i, \ldots, p_n$ by straight movement towards b.

Proof. Given points P in biangular configuration with center b, let r be the minimum distance between a point in P and b. Let $P' = \{p'_1, \ldots, p'_n\}$ be the set of points that arises when we move all points straight in the direction of b until they all have distance r from b (cf. Figure 4). Then all points in P' are on a circle with center b and radius r. Since the angles between adjacent points w.r.t. b are the same in P and P', the points in P' are in biangular configuration as well. Moreover, the points are now rotational symmetric with rotation angle $\frac{2 \cdot 360°}{n} = \alpha + \beta$, and rotation center b. Thus, b is the Weber point of P'. Since P arises from P' by moving each point to its original position, and since all these movements are straight away from b, and since the Weber point is invariant under straight movement, b is the Weber point of P as well.

Since the Weber point is unique (recall that we always assume that the points are not on a line), any point set can have at most one center of biangularity, independent of the angles. On the other hand, any point b defines a unique ordering of all points except for those that are on a same line starting in b (say counterclockwise). Hence the sequence of angles between adjacent points (where some angles might be zero) is unique. □

In our algorithm to find the center of biangularity, if any, we will use *Thales circles* that are defined as follows (refer to Figure 5):

Definition (Thales circle). *Let p and q be two disctinct points on the plane, and α an angle, with $0° < \alpha < 180°$. A circle C is a* **Thales circle of angle** *α for p and q if p and q are on C and there is a point x on C such that $\sphericalangle(p, x, q) = \alpha$.*

The name "Thales circle" refers to Thales of Miletus, who first showed that all angles in a semicircle have 90°. In fact, it is well-known from basic geometry that all angles on one side of a circular segment are equal, i.e., given a Thales circle C of angle α for points p and q, then $\sphericalangle(p, y, q) = \alpha$ for all points y on C that are on the same circular segment of C. Moreover, if $0° < \alpha < 180°$ and C is a Thales circle of angle α for points p and q, then C is a Thales circle of angle $180° - \alpha$ for the two points as well: Let c be the center of C,

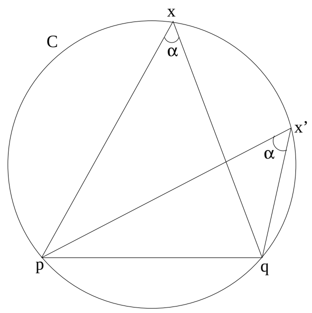

Figure 5: A Thales circle of angle α.

let M be the perpendicular bisector of the line segment from p to q (cf. Figure 6), and let x and y be the point on M that are on C. Since C is a Thales circle, we have angle α in one of these two points, say x. Let $\beta = \sphericalangle(p, y, q)$. Points x and y are opposite points on C. Thus, C is a Thales circle of angle $90°$ for x and y. This yields $\sphericalangle(x, p, y) = 90°$. With $\sphericalangle(p, x, y) = \frac{\alpha}{2}$, we get $\frac{\beta}{2} = \sphericalangle(p, y, x) = 180° - \sphericalangle(x, p, y) - \sphericalangle(p, x, y) = 180° - 90° - \frac{\alpha}{2} = 90° - \frac{\alpha}{2}$. This yields $\beta = 180° - \alpha$.

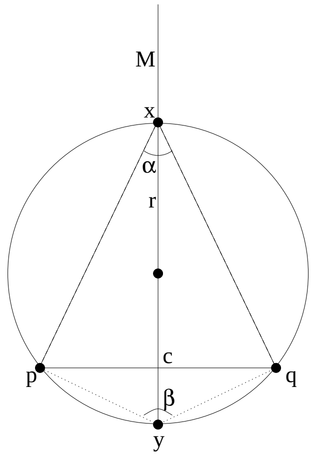

Figure 6: A Thales circle of angle α for p and q is a Thales circle of angle $180° - \alpha$ as well.

Let n points be in equiangular configuration with center e, and let $\alpha = \frac{360°}{n}$. Then there is an ordering of the points such that each two adjacent points have angle α w.r.t. e. Hence, e must lie in the intersection of the corresponding n Thales circles of angle α. This yields a simple (and very inefficient) algorithm to find the center of equiangularity, if it exists: First, we compute all permutations of the points. For each permutation, we compute both Thales circles of angle α for adjacent points. Then we intersect each selection of n of these circles.

If they intersect in exactly one point, then this is the center of equiangularity. If the intersection is empty for all permutations and all selections of Thales circles, then the points are not in equiangular configuration. For biangular configurations, a similar approach is possible, since n points are in biangular configuration if and only if they can be partitioned into two disjoint subsets so that they are in equiangular configuration with the same center.

Obviously, these sketched algorithms have finite, but super-exponential running time. It is possible to improve the algorithms to obtain a polynomial running time, but we abstain from presenting this, since we will give algorithms with linear running time in the Section 3.

We conclude this section with showing how to construct a Thales circle of a specific angle α in constant time:

Lemma 2. *Given two points p, q and an angle $\alpha(0° < \alpha < 180°)$, a Thales circle of angle α for p and q can be constructed in constant time. Moreover, there is exactly one Thales circle of angle $90°$, and there are exactly two Thales circles of angle $\alpha \neq 90°$ for p and q.*

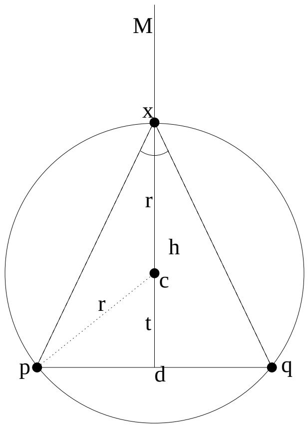

Figure 7: Construction of a Thales circle of angle α

Proof. We first show how to construct a Thales circle of angle α where $\alpha < 90$: Let $d := |p - q|$, L be the line segment from p to q, and M be the "lefthanded" part of the perpendicular bisector of L (cf. Figure 7). Let

$$h := \frac{d}{2 \tan \frac{\alpha}{2}},$$

and let x be the point on M with distance h from L. Then we have $\sphericalangle(p, x, q) = \alpha$. To obtain r, we use the fact that $r = \sqrt{(\frac{d}{2})^2 + t^2}$ and $h = r + t$. This gives us

$$
\begin{aligned}
& & r &= \sqrt{(\frac{d}{2})^2 + (h-r)^2} \\
&\Longrightarrow & r^2 &= (\frac{d}{2})^2 + h^2 - 2hr + r^2 \\
&\Longleftrightarrow & r &= \frac{(\frac{d}{2})^2 + h^2}{2h} \\
&\Longleftrightarrow & r &= \frac{d^2}{8h} + \frac{h}{2}.
\end{aligned}
$$

Let c be the point on M with distance r from x and C be the circle with center c and radius r. Then p, q and x are on C and we have $\sphericalangle(p, x, q) = \alpha$. Thus, C is a Thales circle of angle α.

If $\alpha = 90°$, then the circle C which has p and q on a diameter is a Thales circle. For $\alpha > 90°$, we can use the construction above for a Thales circle of angle $180° - \alpha$. Then Lemma 2 implies that this is already a Thales circle of angle α as well.

For the second claim, observe first that each Thales circle can be flipped along the line segment from p to q. This yields a second Thales circle, except for the case of $\alpha = 90°$. No more Thales circles can exist, since all Thales circles for p and q intersect in the two points, and each circle with radius different from r as constructed above yield different angles. □

3 Linear Algorithms for Equi- and Biangular Configurations

In this section, we present algorithms to find the center of equiangularity or biangularity, if any. For the case of equiangularity, we distinguish two cases depending on the parity of n, the number of points. The algorithm for n even yields immediately an algorithm to find the center of biangularity, if any.

Lemma 3. *Given n points $P = \{p_1, \ldots, p_n\}$, with n even, there is an algorithm with running time linear in n that decides whether the points are in equiangular configuration, and if so, outputs the center of equiangularity.*

Proof. If $n = 2$, then the points are always in equiangular configuration. Hence, we can assume that $n > 2$.

Assume for a moment that the points are in equiangular configuration with center e. For every point $p \in P$ there is exactly one corresponding point $p' \in P$ on the prolongation of the line from p to e (there cannot be more than one point on this line, since this would imply an angle of $0°$ between these points w.r.t. e). Hence, the line through p and p' divides the set of points in two subsets of equal cardinality $n/2 - 1$, and e lies on the intersection of all these lines. This will be used in the following to find e, if it exists.

Given point set P, we first compute a point x on the convex hull of P (observe that the line which divides the set of points in two subsets of equal cardinality is only unique for points on the convex hull). Instead of computing the entire convex hull itself - which would take time $\Theta(n \log n)$ - we take as x any point with minimal x-coordinate. This is a point on the convex hull. Then we compute the slope of the line from x to p_i for all points $p_i \neq x$, and store them (unsorted) in an array. We pick some median element x' in this array of slopes and call the corresponding line from x to x' M_x, the *median line* of x. This takes time linear in n, since selecting the k-th element - and consequently the median as well - in an array can be done in linear time [5]. If there are any other points on the line M_x from x to x', then the points in P cannot be in equiangular configuration. Otherwise, M_x separates the points in P in two subsets of size $\frac{n}{2} - 1$.

We choose a second point y as a point with maximal slope in the array. If there is more than one candidate for y, then we take the one closest to x. Then point y is the neighbour of x on the convex hull (clockwise). Again, we pick some point y' such that the slope of the corresponding line M_y is the median among all slopes; if there are other points on M_y, then again the points in P are not in equiangular configuration. Otherwise, the two median lines M_x and M_y are different: Assume by contradiction that $M_x = M_y$. Then $x' = y$ and $y' = x$, and M_x is the line from x to y. Since x and y are on the convex hull of P, all points from P are on one side of M_x. On the other hand, M_x is a median line, hence the number of points from P on both sides of M_x is equal. Hence, $n = 2$, in contradiction to our assumption at the beginning of this proof.

The two median lines M_x and M_y do intersect (if x has $\frac{n}{2} - 1$ points on the same side as y, then a parallel line to M_x through y has at most $\frac{n}{2} - 2$ points on one side), and their intersection e is the only candidate for a center of equiangularity. Let $\alpha = \frac{360°}{n}$. We fix an arbitrary point $p \in P$ and compute all angles between p and p_j w.r.t. e for all $p_j \neq p$. If any of these angles is not a multiple of α, then the points are not in equiangular configuration. Otherwise, we store the points from P in an array of length $n - 1$, where we store p_j in the array at position k if the angles between p and p_j w.r.t. e is $k \cdot \alpha$ (this resembles a kind of bucket counting sort, see [5]). If there is exactly one point in each position of the array, then the points are in equiangular configuration with center e (and the array already represents the corresponding ordering). Otherwise the points are not in equiangular configuration.

Obviously, this algorithm runs in time linear in n. □

The same algorithm can be used to find a candidate for center of biangularity, since for biangular configurations n is always even. This yields the following

Corollary 4. *Given n points $P = \{p_1, \ldots, p_n\}$, there is an algorithm with running time linear in n that decides whether the points are in biangular configuration, and if so, outputs the center of biangularity.*

We now show how to find the center of equiangularity for the case n odd. This algorithm is more sophisticated than the one for n even, since the concept of median lines has to be relaxed to median cones. However, the main idea remains the same.

Lemma 5. *Given n points $P = \{p_1, \ldots, p_n\}$, with n odd, there is an algorithm with running time linear in n that decides whether the points are in equiangular configuration, and if so, outputs the center of equiangularity.*

Proof. Assume for a moment that the points are in equiangular configuration with center e. For any point $p \in P$, there is no other point on the line from e to p, since n is odd. Hence, this line divides the set of points in two subsets of equal cardinality $\frac{n-1}{2}$. If we pick two points $p_l, p_u \in P$ that are "closest" to this line, in the sense that the slope of the line from p to e is right between the slope of the lines L_p resp. U_p from p to p_l resp. from p to p_u, then these two points define the *median cone* $Cone_p$ starting in p such that the number of points on each side of the cone (the lines L_p resp. U_p inclusive) equals $\frac{n-1}{2}$ (cf. Figure 8). We will use these cones to find the center of equiangularity, if it exists.

Given point set P, we first compute a point x on the convex hull of P, where x is a point with a minimal x-coordinate. Then we compute the slopes of lines from x to any point $p \in P, p \neq x$, and store them in an (unsorted) array. Then we select points x_l and x_u such that the slope of the lines L_x resp. U_x from x to x_l resp. from x to x_u is the $\lfloor n/2 \rfloor$-th resp. the $\lceil n/2 \rceil$-th largest among all computed slopes. These two points can be found in time linear in n [5]. Let $Cone_x$ be the median cone starting in x that is defined by x_l and x_u. Obviously, there is no point from P strictly inside this cone.

Let y be the clockwise neighbour of x on the convex hull (compare to proof of Lemma 3). As before, we construct the cone $Cone_y$ for point y by defining appropriate points y_l, y_u and lines L_y and U_y. Let p be the intersection of U_x and L_y, and let k_{xy} be the number of points from $P - \{x, y\}$ that lie in or on the convex angle in p with edges U_x and L_y (cf. Figure 9). Obviously, if the points in P were in equiangular configuration, then the center of equiangularity e would be strictly inside the intersection of $Cone_x$ and $Cone_y$. Moreover, in the corresponding ordering of the points there would be exactly k_{xy} points between x and y. Hence, the angle between x and y w.r.t. e would be $(k_{xy} + 1) \cdot \alpha$, with $\alpha = \frac{360°}{n}$, and center e would be on one of the two Thales circles of angle $(k_{xy} + 1) \cdot \alpha$ for x and y. Even without knowing e we can define these two Thales circles. Since x and y are points on the convex hull of P and the center of equiangularity must be inside the convex hull, it is obvious on which of the two Thales circles the center would be, if it exists. Hence, we can define $Circle_{xy}$ to be this Thales circle.

We will now choose a third point z on the convex hull such that a center of equiangularity, if any, lies on the intersection of $Circle_{xy}$ and $Circle_{yz}$ (a corresponding Thales circle), and this intersection has at most two points.

For the choice of z we have to be careful. If we simply took z as the next point clockwise on the convex hull (after x and y), it might happen that the corresponding Thales circle $Circle_{yz}$ of angle $(k_{yz}+1) \cdot \alpha$ coincides with $Circle_{xy}$, and consequently there are still infinitely many possible candidates for the center of equiangularity. Therefore, we will choose z such that it is on the convex hull

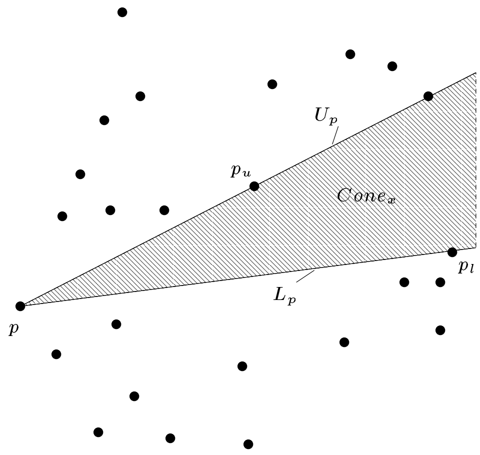

Figure 8: Median Cone $Cone_p$

and not on $Circle_{xy}$ (see fig. 10): Let g be the line through the starting and ending point of C, the circle segment $Circle_{xy} \cap Cone(x) \cap Cone(y)$. Line g defines two halfplanes. If the halfplane which does not contain x and y is empty, then the points are not in equiangular configuration, since a center of equiangularity must be on C, which would be in this case outside the convex hull of P. Let S be the set of points in this halfplane. For each point in S we compute the perpendicular distance to g, and take z as a point with maximal distance. Then z is on the convex hull of P, since all points from P are on one side of the line through z that is parallel to g. Moreover, z is not on $Circle_{xy}$. Analogous to the construction of $Circle_{xy}$ above, we construct a second Thales circle $Circle_{yz}$ of angle $(k_{yz} + 1) \cdot \alpha$.

Since $z \notin Circle_{xy}$, the two circles $Circle_{xy}$ and $Circle_{yz}$ intersect either in one or two points. If they intersect in one point, then they intersect in y, and no center of equiangularity can exist. If they intersect in two points, then one of the two points is y. The other point is the only candidate for a center of equiangularity. Similar to the proof of Lemma 3, it can be checked whether it is a center of equiangularity.

Obviously, the running time of this algorithm is linear in n.

□

4 Conclusion

We have shown that the Weber point of n points in biangular configuration is just the center of biangularity, and that it can be found in linear time. Obviously, this result can be extended to point configurations where the sequence of angles w.r.t. some center is periodic (biangular configuration have period length two, while equiangular configurations have period lenght one). However,

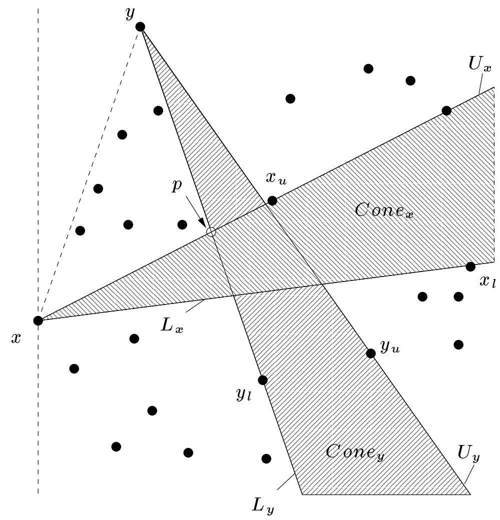

Figure 9: Determination of k_{xy} ($n = 25$)

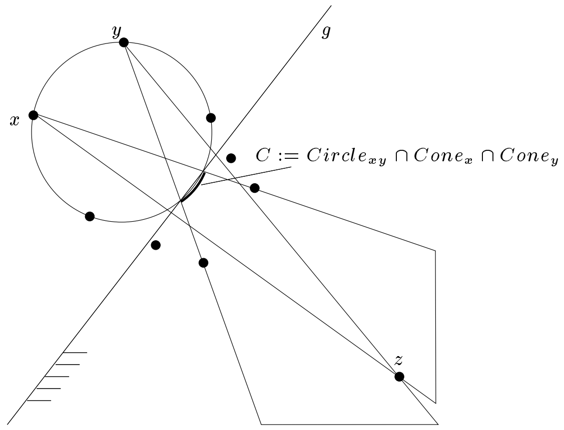

Figure 10: Determination of z

characterizing configurations where the Weber point can always be computed remains an open problem.

Acknowledgements

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